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LETTER TO THE EDITOR

Spacing distributions for some Gaussian ensembles of Hermitian matrices

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Abstract. The probability $E(n, s)$ that an interval of length s contains exactly n eigenvalues of a random matrix is expressed in terms of their correlation functions. For the Gaussian ensemble of Hermitian matrices with an arbitrary ratio of their symmetric and antisymmetric parts, studied earlier, we can thus write $E(n, s)$ as a convergent infinite product multiplied by an infinite sum.

In an earlier paper (Mehta and Pandey 1983) we studied two Gaussian ensembles; one of Hermitian matrices with an arbitrary ratio of their symmetric and antisymmetric parts, and the second of Hermitian quaternion matrices with an arbitrary ratio of their self-dual and anti-self-dual parts. There we stated the reason of our interest and the possible applications. We gave closed expressions for all the correlation and cluster functions of these ensembles.

Here we will express the probability that a given interval contains exactly n eigenvalues in terms of the correlation or the cluster functions, and hence write that probability as an infinite product multiplied by a sum. This study indicates the existence of close relations, yet to be discovered, between prolate spheroidal functions of even and odd orders and their eigenvalues.

Consider an ensemble of $N\times N$ Hermitian matrices $[H_{jk}]=[R_{jk}+\mathrm{i}S_{jk}]$, R and S real, with the joint probability density of the matrix elements as

$$P(H)\propto\exp\left[-\frac{1+\alpha^2}{2}\sum_{j,k}\left(R_{jk}^2+\frac{1}{\alpha^2}S_{jk}^2\right)\right], \tag{1}$$

$$\mathrm{d}H=\prod_{j\leqslant k}\mathrm{d}R_{jk}\prod_{j<k}\mathrm{d}S_{jk}. \tag{2}$$

From the above equations we derived (Mehta and Pandey 1983, Pandey and Mehta 1982) the joint probability density $p(x_1,\ldots,x_N)$ of the eigenvalues $x_1,\ldots,x_N$ of H. The probability of observing n eigenvalues in the intervals $\mathrm{d}x_1,\ldots,\mathrm{d}x_n$ around the points $x_1,\ldots,x_n$ irrespective of the other eigenvalues is $R_n\,\mathrm{d}x_1\ldots\mathrm{d}x_n$, where R_n is the n-level correlation function

$$R_n(x_1,\ldots,x_n)=\frac{N!}{(N-n)!}\int_{-\infty}^{\infty}\ldots\int p(x_1,\ldots,x_N)\,\mathrm{d}x_{n+1}\ldots\mathrm{d}x_N. \tag{3}$$

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The probability that an interval θ contains exactly n eigenvalues is

$$\mathscr{E}(n,\theta)=\frac{N!}{n!(N-n)!}\int_{\text{in}}\cdots\int \mathrm{d}x_1\ldots\mathrm{d}x_n\int_{\text{out}}\cdots\int \mathrm{d}x_{n+1}\ldots\mathrm{d}x_N p(x_1,\ldots,x_N),\qquad(4)$$

where the subscript 'in' means that $x_1,\ldots,x_n$ vary in the interval θ, while the subscript 'out' means that $x_{n+1},\ldots,x_N$ vary outside the interval θ.

Introducing the characteristic function of the interval

$$\chi_\theta(x)=\begin{cases}1 & \text{if } x \text{ lies in } \theta,\\ 0 & \text{otherwise,}\end{cases}\qquad(5)$$

we can write (4) as

$$\mathscr{E}(n,\theta)=\frac{N!}{n!(N-n)!}\int_{-\infty}^{\infty}\cdots\int \mathrm{d}x_1\ldots\mathrm{d}x_N\prod_{i=1}^{n}\chi_\theta(x_i)\prod_{j=n+1}^{N}[1-\chi_\theta(x_j)]p(x_1,\ldots,x_N).\qquad(6)$$

We assume the interval θ to be small so that the level density $R_1(x)$, for x in θ, can be taken to be constant. Measuring θ in terms of the local mean spacing, $s=\theta R_1$, we will write $E(n,s)=\lim\mathscr{E}(n,\theta)$ for $N\to\infty$. This $E(n,s)$ is the probability that an arbitrary interval of length s, measured in units of the local mean spacing, contains exactly n levels. The spacing distributions are related to the second derivatives of the $E(n,s)$; see Mehta and des Cloizeaux (1972).

We will show that

$$E(0,s)=\prod_{j=0}^{\infty}(1-\mu_j),\qquad(7)$$

and for $n>0$

$$E(n,s)=E(0,s)\sum_{0\leq j_1<j_2<\ldots<j_n}\left(\frac{\mu_{j_1}}{1-\mu_{j_1}}\cdots\frac{\mu_{j_n}}{1-\mu_{j_n}}\right),\qquad(8)$$

where $\mu_j\equiv\mu_j(s)$, $j=0,1,2,\ldots$ are the (discrete) eigenvalues of the infinite matrix consisting of the 2×2 blocks

$$\begin{bmatrix}\frac{1}{4}s\gamma_{2l}^2 & \frac{1}{2}sD_{lm}\\ -\frac{1}{2}sJ_{lm} & \frac{1}{4}s\gamma_{2m+1}^2\end{bmatrix}_{l,m=0,1,2,\ldots}\qquad(9)$$

with

$$D_{lm}=\gamma_{2l}\gamma_{2m+1}\int_0^1 x\exp(2\lambda^2\pi^2x^2)f_{2l}(x)f_{2m+1}(x)\,\mathrm{d}x,\qquad(10)$$

$$J_{lm}=\gamma_{2l}\gamma_{2m+1}\int_1^\infty\frac{1}{x}\exp(-2\lambda^2\pi^2x^2)f_{2l}(x)f_{2m+1}(x)\,\mathrm{d}x,\qquad(11)$$

$$\lambda\sqrt{2}=\alpha/(\text{local mean spacing}),\qquad(12)$$

and $f_l(x)\equiv f_l(x;c)$ are the prolate spheroidal functions, solutions of the integral equation

$$\int_{-1}^{1}\mathrm{e}^{\mathrm{i}xyc}f_l(y)\,\mathrm{d}y=\mathrm{i}^l\gamma_l(c)f_l(x),\qquad(13)$$

or of

$$\int_{-1}^{1} \frac{\sin(x-y)c}{(x-y)c} f_l(y)\,\mathrm{d}y = \tfrac{1}{2}\gamma_l^2 f_l(x) \tag{14}$$

normalised as

$$\int_{-1}^{1} f_l(x) f_m(x)\,\mathrm{d}x = \delta_{lm}. \tag{15}$$

The $\gamma_l \equiv \gamma_l(c)$ are the eigenvalues in (13) and

$$c = \tfrac{1}{2}\pi s. \tag{16}$$

One may think that in the limit $N \to \infty$ the result should be the same for any finite value of λ, since it amounts to putting $\alpha = 0$. Actually this limit is delicate and depends on how $\alpha \to 0$.

We now prove (7) and (8). Expanding the product $\Pi[1-\chi_\theta(x_j)]$ in (6) and regrouping similar terms we get

$$\mathscr{E}(n,\theta) = \frac{1}{n!}\sum_{j=n}^{N} \frac{(-1)^{j-n}}{(j-n)!}\int_{-\infty}^{\infty}\dots\int R_j(x_1,\dots,x_j)\prod_{i=1}^{j}\chi_\theta(x_i)\,\mathrm{d}x_i \tag{17}$$

where $R_j(x_1,\dots,x_j)$ is the j-level correlation function, equation (3). Defining

$$r_j = \int_{(x_i \mathrm{in}\,\theta)}\dots\int R_j(x_1,\dots,x_j)\,\mathrm{d}x_1\dots\mathrm{d}x_j, \tag{18}$$

we can write

$$\mathscr{E}(n,\theta) = \frac{(-1)^n}{n!}\sum_{j=n}^{N}\frac{(-1)^j}{(j-n)!} r_j = (\frac{(-1)^n}{n!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \sum_{j=0}^{N}\frac{(-1)^j}{j!} r_j z^j\right]\Bigg|_{z=1}. \tag{19}$$

Introducing the generating function $R(z)$ (Mehta 1967, appendix A.6),

$$R(z) = 1 + \sum_{j=1}^{N}\frac{r_j}{j!} z^j, \tag{20}$$

one can write

$$\mathscr{E}(n,\theta) = \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n R(-z)\Bigg|_{z=1} = \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \exp[T(-z)]\Bigg|_{z=1} \tag{21}$$

where $T(z)$ is the generating function for the cluster functions

$$T(z) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!} t_j z^j, \tag{22}$$

$$t_j = \int_{(x_i \mathrm{in}\,\theta)}\dots\int T_j(x_1,\dots,x_j)\,\mathrm{d}x_1\dots\mathrm{d}x_j. \tag{23}$$

Substituting the expression for T_n obtained by Mehta and Pandey (1983),

$$T_n(x_1,\dots,x_n) = \tfrac{1}{2}\,\mathrm{Tr}\sum \Phi(x_1,x_2)\Phi(x_2,x_3)\dots\Phi(x_n,x_1), \tag{24}$$

in (7) we get

$$t_n = (n-1)!\tfrac{1}{2}\,\mathrm{Tr}\int_{(x_i \mathrm{in}\,\theta)}\dots\int \Phi(x_1,x_2)\Phi(x_2,x_3)\dots\Phi(x_n,x_1)\,\mathrm{d}x_1\dots\mathrm{d}x_n \tag{25}$$

where $\Phi(x, y)$ is a known 2×2 matrix (see equation (2.14) of Mehta and Pandey 1983). We will not repeat its expression here, since it is cumbersome to write it for finite N.

Though not evident, one can convince oneself that the integral equation

$$\int_\theta \Phi(x, y)\mathscr{F}(y)\,\mathrm{d}y = \mu\mathscr{F}(x) \tag{26}$$

has N distinct eigenvalues μ_j, each eigenvalue occurring twice, so that

$$\tfrac{1}{2}\mathrm{Tr}\int_{(x_i \mathrm{in}\,\theta)} \dots \int \Phi(x_1, x_2)\Phi(x_2, x_3)\dots\Phi(x_n, x_1)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \sum_{j=0}^{N-1} \mu_j^n, \tag{27}$$

and from (22) and (25) we have

$$T(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} z^j \sum_i \mu_i^j = \sum_i \ln(1+z\mu_i). \tag{28}$$

Substituting this in (21) we get

$$\mathscr{E}(n, \theta) = \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \prod_i (1-z\mu_i)\bigg|_{z=1}. \tag{29}$$

Taking the limit $N\to\infty$, we can replace Φ by σ (see equation (2.34) of Mehta and Pandey 1983)

$$\sigma(x, y) \equiv \begin{bmatrix} [\sin\pi(x-y)]/\pi(x-y) & D(x-y) \\ J(x-y) & [\sin\pi(x-y)]/\pi(x-y) \end{bmatrix}, \tag{30}$$

$$D(r) = -\frac{1}{\pi}\int_0^\pi t\sin(tr)\exp(2\lambda^2t^2)\,\mathrm{d}t, \tag{31}$$

$$J(r) = -\frac{1}{\pi}\int_\pi^\infty \frac{\sin(tr)}{t}\exp(-2\lambda^2t^2)\,\mathrm{d}t. \tag{32}$$

The integral equation (26) becomes in this limit

$$\int_{-s/2}^{s/2} \sigma(x, y)F(y)\,\mathrm{d}y = \mu F(x) \tag{33}$$

and we have

$$\lim \mathscr{E}(n, \theta) \equiv E(n, s) = \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \prod_{i=0}^{\infty} (1-z\mu_i)\bigg|_{z=1} \tag{34}$$

where μ_i are now the distinct eigenvalues of (33).

Making a change of variables in (33) to bring the limits of integration to $(-1, 1)$ and expressing the unknown solution as a linear combination of prolate spheroidal functions, we can use (13), (14) and (15) to conclude that $2\mu_j/s$ are the eigenvalues of the infinite matrix

$$M_{lm} = \begin{bmatrix} A_{lm} & B_{lm} \\ C_{lm} & A_{lm} \end{bmatrix}_{l,m=0,1,2,\dots}, \tag{35}$$

where

$$A_{lm} = \int_{-1}^{1}\int \mathrm{d}x\,\mathrm{d}y \frac{\sin(x-y)c}{(x-y)c} f_l(x)f_m(y)$$
$$= \tfrac{1}{2}\int\int_{-1}^{1}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\, \mathrm{e}^{\mathrm{i}(x-y)cz} f_l(x)f_m(y)$$
$$= \tfrac{1}{2}\gamma_l^2\delta_{lm}, \tag{36}$$

$$B_{lm} = \int_{-1}^{1}\int \mathrm{d}x\,\mathrm{d}y \int_0^1 \mathrm{d}k\, k \exp(2\lambda^2\pi^2k^2) \sin kc(x-y) f_l(x)f_m(y)$$
$$= \gamma_l\gamma_m \tfrac{1}{2}[(-1)^l-(-1)^m] \int_0^1 \mathrm{d}k\, k\exp(2\lambda^2\pi^2k^2) f_l(k)f_m(k), \tag{37}$$

$$C_{lm} = \int_{-1}^{1}\int \mathrm{d}x\,\mathrm{d}y \int_1^{\infty} \mathrm{d}k \frac{\exp(-2\lambda^2\pi^2k^2)}{k} \sin kc(x-y) f_l(x)f_m(y)$$
$$= \gamma_l\gamma_m \tfrac{1}{2}[(-1)^l-(-1)^m] \int_1^{\infty} \mathrm{d}k \frac{\exp(-2\lambda^2\pi^2k^2)}{k} f_l(k)f_m(k). \tag{38}$$

Note that $B_{lm}=0=C_{lm}$ if $l+m$ is even, and

$$B_{2l,2m+1} = -B_{2m+1,2l} = D_{lm}, \tag{39}$$
$$C_{2l,2m+1} = -C_{2m+1,2l} = J_{lm}, \tag{40}$$

where D_{lm} and J_{lm} are given by (10) and (11). Thus the eigenvalues of $sM/2$ are the eigenvalues of the matrix (9) each repeated twice.

Equation (34) is equivalent to (7) and (8).

(1) When α is fixed and $N\to\infty$, $\lambda\to\infty$, $D(r)\to\infty$ and $J(r)\to 0$, while the product $J(r)D(r)\to 0$. In this case we can therefore replace $J(r)$ and $D(r)$ by zeros. The matrix (9) is then diagonal and has eigenvalues $\mu_i = \frac{1}{4}s\gamma_i^2$. Equations (7) and (8) then agree with the known result for the Gaussian unitary ensemble (Mehta and des Cloizeaux 1972).

(2) For any λ, taking the trace in (9) we have

$$\sum_i \mu_i = \tfrac{1}{4}s\sum_i \gamma_i^2 \tag{41}$$

but from (13)

$$\sum_i \gamma_i^2 = \int_{-1}^{1}\int \mathrm{e}^{\mathrm{i}xyc}\,\mathrm{e}^{-\mathrm{i}yxc}\,\mathrm{d}x\,\mathrm{d}y = 4, \tag{42}$$

so that $\Sigma_i\,\mu_i = s$. This provides a check for any calculation, numerical or otherwise.

(3) For $\alpha=0=\lambda$, one can expand the μ_i in powers of s. Thus

$$\begin{aligned} \mu_0 &= s - \tfrac{1}{36}\pi^2s^2 - \tfrac{1}{1296}\pi^4s^3 + \dots, & \mu_1 &= \tfrac{1}{36}\pi^2s^2 + (\tfrac{1}{1296}\pi^4 - \tfrac{1}{225}\pi^2)s^3 + \dots, \\ \mu_2 &= \tfrac{1}{225}\pi^2s^3 - \dots, & \mu_3 &= \tfrac{1}{19600}\pi^4s^4 + \dots. \end{aligned} \tag{43}$$

For $\lambda \neq 0$, a power series expansion in s is not possible.

(4) The case $\alpha = 0 = \lambda$ corresponds to the Gaussian orthogonal ensemble, where one knows (Mehta and des Cloizeaux 1972) that

$$E(0, s) = \prod_{i=0}^{\infty} (1 - \tfrac{1}{4} s\gamma_{2i}^2), \tag{44}$$

while for $n > 0$,

$$E(2n-1, s) = E(0, s) \sum_{(i)} \frac{s\gamma_{2i_1}^2}{4 - s\gamma_{2i_1}^2} \cdots \frac{s\gamma_{2i_n}^2}{4 - s\gamma_{2i_n}^2} \left(\sum_{j=1}^{n} f_{2i_j}(1) \int_{-1}^{1} f_{2i_j}(x)\,\mathrm{d}x \right), \tag{45}$$

$$E(2n, s) = E(0, s) \sum_{(i)} \frac{s\gamma_{2i_1}^2}{4 - s\gamma_{2i_1}^2} \cdots \frac{s\gamma_{2i_n}^2}{4 - s\gamma_{2i_n}^2} \left(1 - \sum_{j=1}^{n} f_{2i_j}(1) \int_{-1}^{1} f_{2i_j}(x)\,\mathrm{d}x \right). \tag{46}$$

The sums (i) in the last two equations are taken over all integers with $0 \leq i_1 < i_2 < \ldots < i_n$.

These should agree with (7) and (8), giving a set of relations among various prolate spheroidal functions and their eigenvalues, not yet recorded.

(5) For the Gaussian ensemble of Hermitian quaternion matrices with an arbitrary ratio of their self-dual and anti-self-dual parts, (7) and (8) are again valid where μ_i are the distinct eigenvalues of

$$\int_{-s/2}^{s/2} \boldsymbol{\sigma}(x-y) F(y)\,\mathrm{d}y = F(x),$$

and $\boldsymbol{\sigma}$ is given by (2.66)–(2.68) of Mehta and Pandey (1983).

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